A CLASSIFICATION OF MAHONIAN MAJ-INV STATISTICS

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Abstract. Two well-known mahonian statistics on words are the inversion number and the major index. In 1996, Foata and Zeilberger introduced generalizations, parametrized by relations, of these statistics. In this paper, we study the statistics which can be written as a sum of these generalized statistics. This leads to generalizations of some classical results. In particular, we characterize all such statistics which are mahonian.

Keywords: words, permutations, Mahonian statistics, major index, inversion number, graphical major index, graphical inversion number, second fundamental transformation.

MR Subject Classifications: Primary 05A05, 05A15; Secondary 05A30.

1. Introduction and main results

1.1. **Introduction.** Let X be a finite alphabet. Without loss of generality we may assume $X = [r] := \{1, 2, \ldots, r\}$. Two of the most known and studied statistics on words (and permutations) are probably the *inversion number* (inv) and the *major index* (maj). They are defined for words $\mathrm{w} = x_1x_2 \ldots x_n$ with letters in X by

$$\mathrm{inv}(\mathrm{w}) = \sum_{1\leq i<j\leq n} \chi(x_i > x_j) \quad \text{and} \quad \mathrm{maj}(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(x_i > x_{i+1}),$$

where, as usual, " $>$ " is the natural order on X with $r > r-1 > \cdots > 2 > 1$, and $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ otherwise.

The major index, originally called *greater index*, was introduced by MacMahon [8]. As explained by Foata and Krattenthaler (see [3] for a discussion), the origin of the inversion

number is not clear but probably MacMahon [8, 9] was the first to consider inversions of words instead of just permutations.

Let $\mathbf{c} = (c(1), c(2), \ldots, c(r))$ be a sequence of r non-negative integers and let v be the non-decreasing word $v = 1^{c(1)}2^{c(2)} \ldots r^{c(r)}$. We will denote by $\mathcal{R}(v)$ (or by $\mathcal{R}(\mathbf{c})$ if there is no ambiguity) the *rearrangement class* of v, that is, the set of all words that can be obtained by permuting the letters of v. A well-known result of MacMahon states that the major index and the inversion number are equidistributed (i.e. have the same generating function) on each rearrangement class $\mathcal{R}(\mathbf{c})$. More precisely, MacMahon showed that the generating function of the statistics maj and inv on each $\mathcal{R}(\mathbf{c})$ is given by

$$\sum_{\mathrm{w}\in\mathcal{R}(\mathbf{c})} q^{\mathrm{inv(w)}} = \sum_{\mathrm{w}\in\mathcal{R}(\mathbf{c})} q^{\mathrm{maj(w)}} = \begin{bmatrix} c(1)+c(2)+\cdots+c(r) \\ c(1), c(2), \cdots, c(r) \end{bmatrix}_q \tag{1.1}$$

where, as usual in q-theory, the q-multinomial coefficient is given by

$$\begin{bmatrix} n_1+n_2+\cdots+n_k \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q = \frac{[n_1+n_2+\ldots+n_k]_q!}{[n_1]_q![n_2]_q!\cdots[n_k]_q!},$$

and the q-factorial $[n]_q!$ by $[n]_q! := (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$. In honor of MacMahon, a statistic which is equidistributed with inv (or maj) on each $\mathcal{R}(\mathbf{c})$ is said to be *mahonian*.

In 1996, Foata and Zeilberger [2] introduced natural generalizations of both "inv" and "maj", parametrized by relations, as follows. Recall that a *relation* U on X is a subset of the cartesian product $X \times X$. For $a, b \in X$, if we have $(a, b) \in U$, we say that a is in relation U to b, and we express this also by $a\mathrm{U}b$. For each such relation U, then associate the following statistics defined on each word $\mathrm{w} = x_1 \ldots x_n$ by

$$\mathrm{inv}'_U(\mathrm{w}) = \sum_{1\le i<j\le n} \chi(x_i \mathrm{U} x_j) \quad \text{and} \quad \mathrm{maj}'_U(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(x_i \mathrm{U} x_{i+1}).$$

For instance, where $U = " > "$ is the natural order on X, then $\mathrm{maj}'_> = \mathrm{maj}$ and $\mathrm{inv}'_> = \mathrm{inv}$. The statistics maj'_U and inv'_U are called *graphical major index* and *graphical inversion number* since a relation on X can be represented by a directed graph on X.

MacMahon's result (1.1) motivates Foata and Zeilberger [2] to pose the following question:

For which relations U on X the statistics maj'_U *and* inv'_U *are equidistributed on each rearrangement class $\mathcal{R}(\mathbf{c})$?*

Generalizing MacMahon's result, they have fully characterized such relations. In order to present their result, we first recall the following definition due to Foata and Zeilberger [2].

Definition 1.1. *A relation U on X is said to be* bipartitional *if there exists an ordered partition $(B_1, B_2, \ldots, B_k)$ of X into blocks B_l together with a sequence $(\beta_1, \beta_2, \ldots, \beta_k)$ of 0's and 1's such that $x\mathrm{U}y$ if and only either (1) $x \in B_l$, $y \in B_{l'}$ and $l < l'$, or (2) $x, y \in B_l$ and $\beta_l = 1$.*

In this paper, we will use the following axiomatic characterization of *bipartitional relations* due to Han [4].

Proposition 1.2. A relation U on X is bipartitional if and only if (1) it is transitive, i.e. $x\mathrm{U}y$ and $y\mathrm{U}z$ imply $x\mathrm{U}z$, and (2) for each $x, y, z \in X$, $x\mathrm{U}y$ and $z \not\mathrm{U} y$ imply $x\mathrm{U}z$.

Then Foata and Zeilberger [2, Theorem 2] proved the following.

Theorem A. *Let U be a relation on X. The statistics maj'_U and inv'_U are equidistributed on each rearrangement class $\mathcal{R}(\boldsymbol{c})$ if and only if U is bipartitional.*

In this paper, we are interesting with statistics which are obtained by summing a graphical major index and a graphical inversion number. In order to motivate this work, we present here two such statistics. The first one is the *Rawlings major index*. In [10], Rawlings have introduced statistics, denoted k-maj ($k \geq 1$), which interpolate the major index and the inversion number and defined for words $\mathrm{w} = x_1 \cdots x_n$ with letters in X by

$$k\text{-maj}(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(x_i \geq x_{i+1} + k) + \sum_{1 \leq i < j \leq n} \chi(x_j + k > x_i > x_j).$$

Note that 1-maj = maj while r-maj = inv. Now, if we set

$$U_k = \{(x, y) \in X^2 \,/\, x \geq y + k\} \quad \text{and} \quad V_k = \{(x, y) \in X^2 \,/\, y + k > x > y\},$$

we have k-maj $= \mathrm{maj}'_{U_k} + \mathrm{inv}'_{V_k}$. In [11], Rawlings proved that for each integer $k \geq 1$, k-maj is a mahonian statistic. Since $U_k \cup V_k$ is the natural order ">" on X, Rawlings 's result can be rewritten $\mathrm{maj}'_{U_k} + \mathrm{inv}'_{V_k}$ and $\mathrm{inv}'_{U_k \cup V_k}$ are equidistributed on each rearrangement class.

The second statistic is more recent and defined on words with letters in a different alphabet. Let $\mathcal{A} = \{A_1, A_2, \cdots, A_r\}$ be a collection of non-empty, finite and mutually disjoints sets of non-negative integers. Combining two statistics introduced by Steingrimsson [12], Zeng and the author [6] have defined a statistic, denoted MAJ, on words $\pi = B_1 B_2 \cdots B_k$ with letters in $\mathcal{A}$ by

$$\mathrm{MAJ}(\pi) = \sum_{1 \leq i \leq k-1} i.\chi(\min(B_i) > \max(B_{i+1})) + \sum_{1 \leq i < j \leq k} \chi(\max(B_j) \geq \min(B_i) > \min(B_j)).$$

For instance, if $\pi = \{3, 9\}\,\{2\}\,\{1, 4, 8\}\,\{7\}\,\{5, 6\}$, then $\mathrm{MAJ}(\pi) = (1 + 4) + (2) = 7$. Let $U_{\mathcal{A}}$ and $V_{\mathcal{A}}$ be the relations defined on $\mathcal{A}$ by

$$(B, B') \in U_{\mathcal{A}} \Leftrightarrow \min(B) > \max(B'),$$
$$(B, B') \in V_{\mathcal{A}} \Leftrightarrow \max(B') \geq \min(B) > \min(B').$$

Then we have $\mathrm{MAJ} = \mathrm{maj}'_{U_{\mathcal{A}}} + \mathrm{inv}'_{V_{\mathcal{A}}}$. It was proved in [6, Theorem 3.5] that

$$\sum_{\pi \in \mathcal{R}(A_1 A_2 \cdots A_r)} q^{\mathrm{MAJ}(\pi)} = [r]_q!. \tag{1.2}$$

Since $U_{\mathcal{A}} \cup V_{\mathcal{A}}$ is a total order on $\mathcal{A}$, it follows from (1.1) that the generating function of $\mathrm{inv}'_{U_{\mathcal{A}} \cup V_{\mathcal{A}}}$ on $\mathcal{R}(A_1 A_2 \cdots A_r)$ is also given by the right-hand side of the above identity. It is then natural to ask if $\mathrm{maj}'_{U_{\mathcal{A}}} + \mathrm{inv}'_{V_{\mathcal{A}}}$ and $\mathrm{inv}'_{U_{\mathcal{A}} \cup V_{\mathcal{A}}}$ are equidistributed on each rearrangement class $\mathcal{R}(\mathrm{w})$ for words w with letters in $\mathcal{A}$.

In view of the above two examples, it is natural to ask: *For which relations U and V on X the statistics $\mathrm{maj}'_U + \mathrm{inv}'_V$ and $\mathrm{inv}'_{U \cup V}$ are*

- *equidistributed on each rearrangement class $\mathcal{R}(\boldsymbol{c})$?*

- *mahonian?*

The purpose of this paper is to answer these questions by fully characterizing all such relations U and V on X.

1.2. **Main results.** Denote by X^* the set of all words with letters in X. In order to simplify the readability of the paper, we introduce the following definition.

Definition 1.3. *A statistic* stat *on* X^* *is a* maj-inv statistic *if there exist two relations* U *and* V *on* X *such that* $\mathrm{stat} = \mathrm{maj}'_U + \mathrm{inv}'_V$.

Clearly, the statistics inv, maj and k-maj are maj-inv statistics on X^*, while MAJ is a maj-inv statistic on $\mathcal{A}^*$. In this paper, a kind of relations on X have a great interest for us. We call them the *κ-extensible relations.*

Definition 1.4. *A relation* U *on* X *is said to be* κ-extensible *if there exists a relation* S *on* X *such that (1)* $U \subseteq S$ *and (2) for any* $x, y, z \in X$, $x\mathrm{U}y$ *and* $z \not{U} y \implies x\mathrm{S}z$ *and* $z \not{S} x$.

If a relation S *on* X *satisfies conditions (1) and (2), we say that* S *is a* κ-extension *of* U *on* X.

We give here some examples of κ-extensible relations.

Example 1.1. *(a) Suppose* $X = \{x, y, z\}$ *and* $U = \{(x, y)\}$*. Then,* $S = \{(x, y), (x, z)\}$ *is a κ-extension of* U *on* X.

(b) The natural order ">" is a κ-extension of the relation $U_k = \{(x, y) \in X^2 \,/\, x \geq y+k\}$ *on* X *for any* $k > 0$.

(c) Let $\mathcal{A} = \{A_1, A_2, \cdots, A_r\}$ *be a collection of non-empty and finite subsets of non-negative integers, and let* $U_{\mathcal{A}}$ *and* $S_{\mathcal{A}}$ *be the relations on* $\mathcal{A}$ *defined by* $(B, B') \in U_{\mathcal{A}} \Leftrightarrow \min(B) > \max(B')$ *and* $(B, B') \in S_{\mathcal{A}} \Leftrightarrow \min(B) > \min(B')$*. Then one can check that* $S_{\mathcal{A}}$ *is a κ-extension of* $U_{\mathcal{A}}$ *on* $\mathcal{A}$.

(d) Every total order is a κ-extension of itself.

In fact the notion of κ-extensible relation can be viewed, by means of the following result, as a generalization of the notion of bipartitional relation.

Proposition 1.5. *A relation* U *on* X *is bipartitional if and only if it is a κ-extension of itself.*

Proof. Using Proposition 1.2, it suffices to see that a relation U is transitive if and only if for any $x, y, z \in X$, $x\mathrm{U}y$ and $z \not{U} y$ imply $z \not{U} x$. Suppose U is transitive and let x, y, z satisfying $x\mathrm{U}y$ and $z \not{U} y$. Suppose $z\mathrm{U}x$, then since $x\mathrm{U}y$, we have by transitivity $z\mathrm{U}y$ which contradict $z \not{U} y$. Thus $z \not{U} x$. Reversely, suppose that $x\mathrm{U}y$ and $z \not{U} y$ imply $z \not{U} x$ for each x, y, z. Let x_1, x_2, x_3 verifying $x_1\mathrm{U}x_2$ and $x_2\mathrm{U}x_3$. Suppose $x_1 \not{U} x_3$. Since $x_2\mathrm{U}x_3$, it then follows that $x_1 \not{U} x_2$ which is impossible. Thus $x_1\mathrm{U}x_3$ and U is transitive. $\square$

We can now present the key result of the paper, which is a generalization of Theorem A.

Theorem 1.6. *Let* U *and* S *be two relations on* X*. The following conditions are equivalent.*

(i) *The statistics* $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$ *and* inv'_S *are equidistributed on each rearrangement class* $\mathcal{R}(\boldsymbol{c})$.

(ii) *S is a κ-extension of U.*

Let U and V be two non-disjoint relations on X and let $(x,y) \in U \cap V$. By definition, $(\mathrm{maj}'_U + \mathrm{inv}'_V)(xy) = 1 + 1 = 2 > 1 \geq \mathrm{inv}'_{U\cup V}(x_1x_2)$ for any $x_1, x_2 \in X$. It follows that if $U \cap V \neq \emptyset$, the statistics $\mathrm{maj}'_U + \mathrm{inv}'_V$ and $\mathrm{inv}'_{U\cup V}$ are not equidistributed on $\mathcal{R}(xy)$. We then obtain immediately from Theorem 1.6 the following result.

Theorem 1.7. *Let U and V be two relations on X. The following conditions are equivalent.*

(i) *The statistics $\mathrm{maj}'_U + \mathrm{inv}'_V$ and $\mathrm{inv}'_{U\cup V}$ are equidistributed on each rearrangement class $\mathcal{R}(\boldsymbol{c})$.*
(ii) *$U \cap V = \emptyset$ and $\mathrm{U} \cup V$ is a κ-extension of U.*

Next, by noting that for a relation S on X, the graphical inversion number inv'_S is mahonian if and only if S is a total order on X, we have obtained the following characterization of mahonian maj-inv statistics.

Theorem 1.8 (Classification of mahonian maj-inv statistics I)**.** *The mahonian maj-inv statistics on X^* are exactly those which can be written $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$, where U and S satisfy the following conditions:*

- *S is a total order on X,*
- *S is a κ-extension of U.*

Moreover, two mahonian maj-inv statistics $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$ and $\mathrm{maj}'_V + \mathrm{inv}'_{T\setminus V}$ are equal on X^* if and only if $S = T$ and $U = V$.

Example 1.2. *(a) It follows from Example 1.1(b) and the above theorem that the statistics k-maj, $k \geq 1$, are mahonian, which was first proved by Rawlings* [11].

(b) Let $\mathcal{A} = \{A_1, A_2, \cdots, A_r\}$ be a collection of nonempty and finite subsets of nonnegative integers, and let $U_{\mathcal{A}}$ and $S_{\mathcal{A}}$ be the relations on $\mathcal{A}$ defined as in Example 1.1(c). It then follows from the above theorem and Example 1.1(c) that MAJ *is mahonian on $\mathcal{A}^*$, which is a generalization of* (1.2).

In fact, we have obtained more precise results on mahonian maj-inv statistics on X^*. Indeed, given a total order S on X, we have characterized all κ-extensible relations U such that S is a κ-extension of U(see Proposition 6.3). As consequence, we have obtained the following result.

Theorem 1.9 (Classification of mahonian maj-inv statistics II)**.** *The mahonian maj-inv statistics on X^* are exactly the statistics $\mathrm{stat}_{f,g}$ defined for words $\mathrm{w} = x_1 \cdots x_n \in X^*$ by*

$$\mathrm{stat}_{f,g}(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(\, f(x_i) \geq g(f(x_{i+1}))\,) + \sum_{1\leq i<j\leq n} \chi(\, g(f(x_j)) > f(x_i) > f(x_j)\,),$$

with f a permutation of X and $g : X \mapsto X \cup \{\infty\}$ a map satisfying $g(y) > y$ for each $y \in X$.

Taking $f = Id$, where Id is the identity permutation, we obtain the following.

Corollary 1.10. *The statistics* stat_g *defined for* $\mathrm{w} = x_1 \cdots x_n \in X^*$ *by*

$$\mathrm{stat}_g(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(\, x_i \geq g(x_{i+1})\,) + \sum_{1\leq i<j\leq n} \chi(\, g(x_j) > x_i > x_j\,),$$

with $g : X \mapsto X \cup \{\infty\}$ *satisfying* $g(y) > y$ *for each* $y \in X$*, are mahonian.*

For instance, the Rawlings major index k-maj is obtained by taking in the previous result $g : X \mapsto X \cup \{\infty\}$ defined by $g(x) = x + k$ if $x + k \leq r$ and $g(x) = \infty$ otherwise.

It is then easy to enumerate the mahonian maj-inv statistics on X^*. Since there are exactly $|X|!$ maps $g : X \mapsto X \cup \{\infty\}$ satisfying $g(y) > y$, we have the following result.

Corollary 1.11. *For each total order* S *on* X*, there are exactly* $|X|!$ *mahonian maj-inv statistics on* X^* *which can be written* $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$.

The paper is organized as follows. In section 2 and section 3, we prove Theorem 1.6. In section 4, we prove Theorem 1.8. In section 5, we characterize all κ-extensible relations on X and prove Theorem 1.9 in section 6. Finally, in section 7, we apply the results of this paper to give new original mahonian statistics on permutations and words.

Remark 1.12. *As pointed by an anonymous referee, some proofs (for instance the proof of the "only if" part of Theorem 1.6) presented in the paper have "simpler proofs" by using a computer algebra system (see e.g.* [5]*).*

2. Proof of the 'if' part of Theorem 1.6

The first direct combinatorial proof of MacMahon's result on the equidistribution of the statistics maj and inv, that is a bijection which sends each word to another one in such a way that the major index of the image equals the number of inversions of the original, is due to Foata [1].

Let U be a κ-extensible relation on X. In this section, we adapt Foata's map, also called *second fundamental transformation* (see e.g. [7]), to construct a bijection Ψ^U of each rearrangement class onto itself such that for each κ-extension S of U, we have

$$\mathrm{inv}'_S(\Psi^U(\mathrm{w})) = (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}). \tag{2.1}$$

2.1. **Notations.** The *length* of a word $\mathrm{w} \in X^*$, denoted by $\lambda(\mathrm{w})$, is its number of letters. By convention, there is an unique word of length 0, the *empty word* ϵ. If Y and Z are subsets of X^*, we designate by YZ the set of words $\mathrm{w} = \mathrm{w}'\mathrm{w}''$ with $\mathrm{w}' \in Y$ and $\mathrm{w}'' \in Z$.

Each $x \in X$ determines a partition of X in two subsets L_x and R_x as follows: the set R_x is formed with all $y \in X$ such that $y\mathrm{U}x$, while the set L_x is formed with all $y \in X$ such that $y \not\mathrm{U} x$.

2.2. **The map** Ψ^U**.** Let w be a word in X^* and $x \in X$. If $\mathrm{w} = \epsilon$, we set $\gamma_x^U(\mathrm{w}) = \epsilon$. Otherwise two cases are to be considered:

(i) the last letter of w is in R_x,
(ii) the last letter of w is in L_x.

Let $(\mathrm{w}_1x_1, \mathrm{w}_2x_2, \ldots, \mathrm{w}_hx_h)$ be the factorization of w having the following properties:

- In case (i) $x_1, x_2, \ldots, x_h$ are in R_x and $\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_h$ are words in L_x^*.

- In case (ii) $x_1, x_2, \ldots, x_h$ are in L_x and $\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_h$ are words in R_x^*.

Call *x-factorization* the above factorization. Clearly, each word has an unique x-factorization. In both cases we have $\mathrm{w} = \mathrm{w}_1 x_1 \mathrm{w}_2 x_2 \ldots \mathrm{w}_s x_s$, then define

$$\gamma_x^U(\mathrm{w}) = x_1 \mathrm{w}_1 x_2 \mathrm{w}_2 \ldots x_s \mathrm{w}_s.$$

The map Ψ^U is then defined by induction on the length of words in the following way:

$$\Psi^U(\epsilon) = \epsilon\ , \tag{2.2}$$

$$\Psi^U(\mathrm{w}x) = \gamma_x^U(\Psi^U(\mathrm{w}))\, x \quad \text{for all } x \in X \text{ and } \mathrm{w} \in X^*. \tag{2.3}$$

Note that Foata's map correspond to the case $U :=$ " $>$ " is the natural order.

Theorem 2.1. *The map Ψ^U is a bijection of X^* onto itself such that for each $\mathrm{w} \in X^*$, we have $\Psi^U(\mathrm{w}) \in \mathcal{R}(\mathrm{w})$, both w and $\Psi^U(\mathrm{w})$ end with the same letter and for each κ-extension S of U, we have*

$$\mathrm{inv}'_S(\Psi^U(\mathrm{w})) = (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}). \tag{2.4}$$

The proof of the above theorem is very similar to the proof in [1, 7]. It is based on the following lemma. Let S be a κ-extension of U. For each $\mathrm{w} = x_1 \ldots x_n \in X^*$, denote by $l_x(\mathrm{w})$ (resp. $r_x(\mathrm{w})$) the number of subscripts j for which $x_j \in L_x$ (resp. $x_j \in R_x$) and $t_x(\mathrm{w})$ designate the number of subscripts j such that $x_j \not{U} x$ and x_jSx. Note that we always have $l_x(\mathrm{w}) + r_x(\mathrm{w}) = \lambda(\mathrm{w})$ and $r_x(\mathrm{w}) + t_x(\mathrm{w})$ is the number of subscripts j for which x_jSx.

Lemma 2.2. *For each $\mathrm{w} \in X^*$ and $x \in X$, the following identities hold:*

$$\begin{aligned}
\mathrm{inv}'_S(\mathrm{w}x) &= \mathrm{inv}'_S(\mathrm{w}) + r_x(\mathrm{w}) + t_x(\mathrm{w}), && (2.5)\\
\mathrm{inv}'_S(\gamma_x^U(\mathrm{w})) &= \mathrm{inv}'_S(\mathrm{w}) - r_x(\mathrm{w}) \qquad \textit{if } \mathrm{w} \in X^* L_x, && (2.6)\\
\mathrm{inv}'_S(\gamma_x^U(\mathrm{w})) &= \mathrm{inv}'_S(\mathrm{w}) + l_x(\mathrm{w}) \qquad \textit{if } \mathrm{w} \in X^* R_x, && (2.7)\\
(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}x) &= (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}) + t_x(\mathrm{w}) \qquad \textit{if } \mathrm{w} \in X^* L_x, && (2.8)\\
(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}x) &= (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}) + t_x(\mathrm{w}) + \lambda(\mathrm{w}) \quad \textit{if } \mathrm{w} \in X^* R_x. && (2.9)
\end{aligned}$$

Proof. By definition, we have the following identities:

$$\begin{aligned}
\mathrm{inv}'_U(\mathrm{w}x) &= \mathrm{inv}'_U(\mathrm{w}) + r_x(\mathrm{w})\\
\mathrm{inv}'_{S\setminus U}(\mathrm{w}x) &= \mathrm{inv}'_{S\setminus U}(\mathrm{w}) + t_x(\mathrm{w})\\
\mathrm{maj}'_U(\mathrm{w}x) &= \mathrm{maj}'_U(\mathrm{w}) \qquad \text{if } \mathrm{w} \in X^* L_x,\\
\mathrm{maj}'_U(\mathrm{w}x) &= \mathrm{maj}'_U(\mathrm{w}) + \lambda(\mathrm{w}) \quad \text{if } \mathrm{w} \in X^* R_x,
\end{aligned}$$

from which we derive immediately (2.8) and (2.9). To obtain (2.5), it suffices to note that $\mathrm{inv}'_S = \mathrm{inv}'_U + \mathrm{inv}'_{S\setminus U}$ (since $U \cap (S \setminus U) = \emptyset$ and $U \subseteq S$). It remains to prove (2.6) and (2.7).

Suppose $\mathrm{w} \in X^* L_x$ and let $(\mathrm{w}_1 x_1, \mathrm{w}_2 x_2, \ldots, \mathrm{w}_s x_s)$ be the x-factorization of w. First, assume that

$$\mathrm{inv}'_S(x_i \mathrm{w}_i) = \mathrm{inv}'_S(\mathrm{w}_i x_i) - \lambda(\mathrm{w}_i) \quad \text{for } 1 \le i \le s. \tag{2.10}$$

Since $\gamma_x(\mathrm{w}) = x_1\mathrm{w}_1x_2\mathrm{w}_2\cdots x_h\mathrm{w}_h$, it is not hard to see that $\mathrm{inv}'_S(\gamma_x(\mathrm{w}))$ is equal to $\mathrm{inv}'_S(\mathrm{w}x)$ decreased by $\lambda(\mathrm{w}_1)+\lambda(\mathrm{w}_2)+\cdots+\lambda(\mathrm{w}_s)$. Since $s = l_x(\mathrm{w})$, we get

$$\mathrm{inv}'_S(\gamma_x(\mathrm{w})) = \mathrm{inv}'_S(\mathrm{w}x) - (\lambda(\mathrm{w}) - s) = \mathrm{inv}'_S(\mathrm{w}x) - r_x(\mathrm{w}),$$

which is exactly (2.6). We now prove (2.10). Let $\tau = \tau_1\tau_2\cdots\tau_m \in R_x^*$ and $y \in L_x$. By definition, we have $\tau_i \mathrm{U} x$ for each i and $y \not\mathrm{U} x$. Since S is a κ-extension of U, it follows that for each i, $\tau_i \mathrm{S} \mathrm{y}$ and $y \not\mathrm{S} \tau_i$. We then have $\mathrm{inv}'_S(y\tau) = \mathrm{inv}'_S(\tau)$ and $\mathrm{inv}'_S(\tau y) = \mathrm{inv}'_S(\tau)+m = \mathrm{inv}'_S(y\tau)+\lambda(\tau)$. Equation (2.10) is obtained by noting that in the x-factorization of $\mathrm{w} \in X^*L_x$, the words $\mathrm{w}_1,\ldots,\mathrm{w}_h$ are in R_x^* and the letters $x_1,\ldots,x_h$ are in L_x.

Equation (2.7) has an analogous proof. Suppose $\mathrm{w} \in X^*R_x$ and let $(\mathrm{w}_1x_1, \mathrm{w}_2x_2, \ldots, \mathrm{w}_hx_h)$ be the x-factorization of w. First, assume that

$$\mathrm{inv}'_S(x_i\mathrm{w}_i) = \mathrm{inv}'_S(\mathrm{w}_ix_i) + \lambda(\mathrm{w}_i) \quad \text{for } 1 \le i \le h. \tag{2.11}$$

Since $\gamma_x(\mathrm{w}) = x_1\mathrm{w}_1x_2\mathrm{w}_2\cdots x_h\mathrm{w}_h$, it is not hard to see that $\mathrm{inv}'_S(\gamma_x(\mathrm{w}))$ is equal to $\mathrm{inv}'_S(\mathrm{w}x)$ increased by $\lambda(\mathrm{w}_1)+\lambda(\mathrm{w}_2)+\cdots+\lambda(\mathrm{w}_s)$. Since $h = r_x(\mathrm{w})$, we get

$$\mathrm{inv}'_S(\gamma_x^U(\mathrm{w})) = \mathrm{inv}'_S(\mathrm{w}x) - (\lambda(\mathrm{w}) - h) = \mathrm{inv}'_S(\mathrm{w}x) + l_x(\mathrm{w}),$$

which is exactly (2.7). It then remains to prove (2.11). Let $\tau = \tau_1\tau_2\cdots\tau_m \in L_x^*$ and $y \in R_x$. By definition, we have $y\mathrm{U}x$ and $\tau_i \not\mathrm{U} x$ for each i. Since S is a κ-extension of U, it follows that for each i, $y\mathrm{S}\tau_\mathrm{i}$. It is then easy to obtain $\mathrm{inv}'_S(\tau y) = \mathrm{inv}'_S(\tau)$ and $\mathrm{inv}'_S(y\tau) = \mathrm{inv}'_S(\tau)+m = \mathrm{inv}'_S(\tau y)+m$. Equation (2.11) is obtained by noting that in the x-factorization of $\mathrm{w} \in X^*R_x$, the words $\mathrm{w}_1,\ldots,\mathrm{w}_h$ are in L_x^* and the letters $x_1,\ldots,x_h$ are in R_x.

□

Proof of Theorem 2.1: By construction, both w and $\Psi^U(\mathrm{w})$ end with the same letter. Let X_n be the set of words in X^* with length n. It is sufficient to verify by induction on n that for all $n \ge 0$, the restriction Ψ_n^U of Ψ^U to X_n is a permutation of X_n satisfying: for any $\mathrm{w} \in X_n$,

$$\Psi_n^U(\mathrm{w}) \in \mathcal{R}(\mathrm{w}) \quad \text{and} \quad \mathrm{inv}'_S(\Psi_n^U(\mathrm{w})) = (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}). \tag{2.12}$$

Since the induction is based on Lemma 2.2 and is very similar to the proof concerning the second fundamental transformation, we refer the reader to [1, 7]. □

3. Proof of the 'only if' part of Theorem 1.6

Let U and S be two relations on X such that the statistics $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$ and inv_S are equidistributed on each rearrangement class $\mathcal{R}(\mathrm{w})$, $\mathrm{w} \in X^*$. We prove here that this imply that S is a κ-extension of U.

3.1. **The relation U is contained in S.** By definition of the graphical statistics, we have that for all word w of length 2, $\mathrm{maj}'_U(\mathrm{w}) = \mathrm{inv}'_U(\mathrm{w})$. Moreover, for each pair (A,B) of disjoints relations, we have $\mathrm{inv}'_A + \mathrm{inv}'_B = \mathrm{inv}'_{A\cup B}$. It then follows that for all $\mathrm{w} \in X^*$, $\lambda(\mathrm{w}) = 2$,

$$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}) = (\mathrm{inv}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w}) = \mathrm{inv}'_{S\cup U}(\mathrm{w}) = (\mathrm{inv}'_S + \mathrm{inv}'_{U\setminus S})(\mathrm{w}).$$

The equidistribution of $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$ and inv'_S on each $\mathcal{R}(\mathrm{w})$, $\lambda(\mathrm{w}) = 2$, then implies that $\mathrm{inv}'_{U\setminus S}(\mathrm{w}) = 0$ for all $\mathrm{w} \in X^*$, $\lambda(\mathrm{w}) = 2$, and thus, $U \setminus S = \emptyset$, i.e., $U \subseteq S$.

3.2. **For any $x, y, z \in X$, $x\mathrm{U}y$ and $z \not\mathrm{U} y$ imply $x\mathrm{Sz}$ and $z \not\mathrm{S} x$.** To simplify the readability of the rest of the proof, we set $V := S \setminus U$, i.e. $U \cap V = \emptyset$ and $U \cup V = S$. In particular, for any $x_1, x_2 \in X$,

$$\chi(x_1\mathrm{Sx_2}) = \chi(\mathrm{x_1Ux_2}) + \chi(\mathrm{x_1Vx_2}) \quad \text{and} \quad \chi(\mathrm{x_1Ux_2}).\chi(\mathrm{x_1Vx_2}) = 0. \tag{3.1}$$

Let $x, y, z \in X$ verifying $x\mathrm{U}y$ and $z \not\mathrm{U} y$. First, note that x and z are distinct, otherwise we have $x\mathrm{U}y$ and $x \not\mathrm{U} y$. Thus $x \neq z$.

3.2.1. *The case $x = y$.* We then have $x\mathrm{U}x$ and $z \not\mathrm{U} x$ and thus $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxx) = \chi(z\mathrm{U}x) + 2\chi(x\mathrm{U}x) + 2\chi(z\mathrm{V}x) + \chi(x\mathrm{V}x) = 2 + 2\chi(z\mathrm{V}x)$. Since $\mathrm{inv}'_S(\mathrm{w}) \leq 3$ for each word w of length 3, it follows that $\chi(z\mathrm{V}x) = 0$, i.e. $z \not\mathrm{V} x$. But $z \not\mathrm{U} x$ and thus $z \not\mathrm{S} x$. Now, suppose $x \not\mathrm{S} z$. It follows that $\mathrm{inv}'_S(xxz) = \mathrm{inv}'_S(xzx) = \mathrm{inv}'_S(zxx) = 1 < 2 = (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxx)$, which contradict the equidistribution of our two statistics on $\mathcal{R}(x^2z)$. Thus we have $x\mathrm{Sz}$ and $z \not\mathrm{S} x$ as desired.

3.2.2. *The case $x \neq y$.* Two cases are to be considered.

Suppose $y = z$. We then have $x\mathrm{U}z$ and $z \not\mathrm{U} z$. Since $U \subseteq S$, we have $x\mathrm{Sz}$. It then suffices to show that $z \not\mathrm{S} x$. Suppose $z\mathrm{Sx}$. We then have

$$\begin{aligned}
(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxz) &= \chi(z\mathrm{U}x) + 2\chi(x\mathrm{U}z) + \chi(z\mathrm{V}x) + \chi(z\mathrm{V}z) + \chi(x\mathrm{V}z) \\
&= 2 + \chi(z\mathrm{U}x) + \chi(z\mathrm{V}x) + \chi(z\mathrm{V}z) = 2 + \chi(z\mathrm{Sx}) + \chi(\mathrm{zVz}) \\
&= 3 + \chi(z\mathrm{V}z),
\end{aligned}$$

and thus $z \not\mathrm{S} z$. Then, it is not hard to see that this imply that $\mathrm{inv}'_S \leq 2$ on $\mathcal{R}(xz^2)$ which, considering $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxz) =\ 3$, contradict the equidistribution of our statistics on $\mathcal{R}(xz^2)$. It follows that $z \not\mathrm{S} x$ as desired. It then remains to consider the last case.

Suppose $y \neq z$. Then x, y, z *are three distinct elements* satisfying $x\mathrm{U}y$ and $z \not\mathrm{U} y$. The next table gives the distribution of $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$ and inv'_S on $\mathcal{R}(xyz)$ after some simplifications obtained by using (3.1).

w	$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w})$	$\mathrm{inv}'_S(\mathrm{w})$
xyz	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{yUz}) + \chi(\mathrm{xVz})$	$1 + \chi(x\mathrm{Sz}) + \chi(\mathrm{ySz})$
xzy	$\chi(x\mathrm{Sz}) + \chi(\mathrm{zVy})$	$1 + \chi(x\mathrm{Sz}) + \chi(\mathrm{zVy})$
yxz	$\chi(y\mathrm{Sx}) + \chi(\mathrm{xSz}) + \chi(\mathrm{xUz}) + \chi(\mathrm{yVz})$	$\chi(y\mathrm{Sx}) + \chi(\mathrm{ySz}) + \chi(\mathrm{xSz})$
yzx	$\chi(y\mathrm{Sz}) + \chi(\mathrm{zSx}) + \chi(\mathrm{zUx}) + \chi(\mathrm{yVx})$	$\chi(y\mathrm{Sz}) + \chi(\mathrm{ySx}) + \chi(\mathrm{zSx})$
zxy	$2 + \chi(z\mathrm{Sx}) + \chi(\mathrm{zVy})$	$1 + \chi(z\mathrm{Sx}) + \chi(\mathrm{zVy})$
zyx	$\chi(y\mathrm{Sx}) + \chi(\mathrm{yUx}) + \chi(\mathrm{zVy}) + \chi(\mathrm{zVx})$	$\chi(z\mathrm{V}y) + \chi(z\mathrm{Sx}) + \chi(\mathrm{ySx})$

(a) Suppose $x\mathrm{Sz}$ and $z\mathrm{Sx}$. We then have $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxy) = 3 + \chi(z\mathrm{V}y)$ and since $\mathrm{inv}'_S \leq 3$ on $\mathcal{R}(xyz)$, we have $z \not\mathrm{V} y$ and thus $z \not\mathrm{S} y$. Using identities $x\mathrm{Sz}$, $z\mathrm{Sx}$ and $z \not\mathrm{S} y$, we obtain the following table

w	$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w})$	$\mathrm{inv}'_S(\mathrm{w})$
xyz	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{yUz}) + \chi(\mathrm{xVz})$	$2 + \chi(y\mathrm{Sz})$
xzy	1	2
yxz	$1 + \chi(y\mathrm{Sx}) + \chi(\mathrm{xUz}) + \chi(\mathrm{yVz})$	$1 + \chi(y\mathrm{Sx}) + \chi(\mathrm{ySz})$
yzx	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{zUx}) + \chi(\mathrm{yVx})$	$1 + \chi(y\mathrm{Sx}) + \chi(\mathrm{ySz})$
zxy	3	2
zyx	$\chi(y\mathrm{Sx}) + \chi(\mathrm{yUx}) + \chi(\mathrm{zVx})$	$1 + \chi(y\mathrm{Sx})$

which imply that $y \not\mathrm{S} x$ (otherwise, $\mathrm{inv}'_S \geq 2$ on $\mathcal{R}(xyz)$ and $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(xzy) = 1$, which is impossible) and thus, by using $y \not\mathrm{S} x$, we get

w	$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w})$	$\mathrm{inv}'_S(\mathrm{w})$
xyz	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{yUz}) + \chi(\mathrm{xVz})$	$2 + \chi(y\mathrm{Sz})$
xzy	1	2
yxz	$1 + \chi(x\mathrm{U}z) + \chi(y\mathrm{V}z)$	$1 + \chi(y\mathrm{Sz})$
yzx	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{zUx}) + \chi(\mathrm{yVx})$	$1 + \chi(y\mathrm{Sz})$
zxy	3	2
zyx	$\chi(z\mathrm{V}x)$	1

Since $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxy) = 3$, we have by equidistribution of our two statistics, $y\mathrm{Sz}$. It follows that zyx is the unique world in $\mathcal{R}(xyz)$ for which $\mathrm{inv}'_S(zyx) = 1$, while $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zyx) \leq (\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(xzy) \leq 1$, which contradict the equidistribution of our two statistics on $\mathcal{R}(xyz)$.

(b) Suppose $x \not\mathrm{S} z$ and $z\mathrm{Sx}$. By a similar reasoning than in (a), we have $z \not\mathrm{V} y$ and thus $z \not\mathrm{S} y$, which lead to the following table.

w	$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w})$	$\mathrm{inv}'_S(\mathrm{w})$
xyz	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{yUz})$	$1 + \chi(y\mathrm{Sz})$
xzy	0	1
yxz	$\chi(y\mathrm{Sx}) + \chi(\mathrm{yVz})$	$\chi(y\mathrm{Sx}) + \chi(\mathrm{ySz})$
yzx	$1 + \chi(y\mathrm{Sz}) + \chi(\mathrm{zUx}) + \chi(\mathrm{yVx})$	$1 + \chi(y\mathrm{Sx}) + \chi(\mathrm{ySz})$
zxy	3	2
zyx	$\chi(y\mathrm{Sx}) + \chi(\mathrm{yUx}) + \chi(\mathrm{zVx})$	$1 + \chi(y\mathrm{Sx})$

Since $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(zxy) = 3$, we must have $y\mathrm{Sz}$ and $y\mathrm{Sx}$, which imply that $\mathrm{inv}'_S \geq 1$ on $\mathcal{R}(xyz)$, which is impossible since $(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(xzy) = 0$.

(c) Suppose $x \not\mathrm{S} z$ and $z \not\mathrm{S} x$. We then get the following table.

w	$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w})$	$\mathrm{inv}'_S(\mathrm{w})$
xyz	$1+\chi(y\mathrm{Sz})+\chi(\mathrm{yUz})$	$1+\chi(y\mathrm{Sz})$
xzy	$\chi(z\mathrm{V}y)$	$1+\chi(z\mathrm{V}y)$
yxz	$\chi(y\mathrm{Sx})+\chi(\mathrm{yVz})$	$\chi(y\mathrm{Sx})+\chi(\mathrm{ySz})$
yzx	$\chi(y\mathrm{Sz})+\chi(\mathrm{yVx})$	$\chi(y\mathrm{Sx})+\chi(\mathrm{ySz})$
zxy	$2+\chi(z\mathrm{V}y)$	$1+\chi(z\mathrm{V}y)$
zyx	$\chi(y\mathrm{Sx})+\chi(\mathrm{yUx})+\chi(\mathrm{zVy})$	$\chi(z\mathrm{V}y)+\chi(y\mathrm{Sx})$

It then follows that $\mathrm{inv}'_S \le 2$ on $\mathcal{R}(xyz)$, and thus, by considering $(\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U})(zxy)$ and $(\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U})(xyz) = 1+2\chi(y\mathrm{U}z)+\chi(y\mathrm{V}z)$, we must have $y\,\not{\mathrm{U}}z$ and $z\,\not{\mathrm{V}}y$, which lead to the following table.

w	$(\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U})(\mathrm{w})$	$\mathrm{inv}'_S(\mathrm{w})$
xyz	$1+\chi(y\mathrm{V}z)$	$1+\chi(y\mathrm{V}z)$
xzy	0	1
yxz	$\chi(y\mathrm{Sx})+\chi(\mathrm{yVz})$	$\chi(y\mathrm{Sx})+\chi(\mathrm{yVz})$
yzx	$\chi(y\mathrm{V}z)+\chi(y\mathrm{V}x)$	$\chi(y\mathrm{Sx})+\chi(\mathrm{yVz})$
zxy	2	1
zyx	$\chi(y\mathrm{Sx})+\chi(\mathrm{yUx})$	$\chi(y\mathrm{Sx})$

Since $(\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U})(xzy) = 0$, it follows that $y\,\not{\mathrm{S}}x$, and thus $\mathrm{inv}'_S \le 1$ on $\mathcal{R}(xyz)$, which is impossible since $(\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U})(zxy) = 2$.

(d) Finally, we have $x\mathrm{Sz}$ and $z\,\not{\mathrm{S}}x$, and thus S is a κ-extension of U. This conclude the proof of the 'only if' part of Theorem 1.6.

4. Mahonian maj-inv statistics

This section is dedicated to the proof of Theorem 1.8. We begin with two lemmas.

Lemma 4.1. *Let S be a relation on X. Then inv'_S is mahonian on X^* if and only if S is a total order.*

Lemma 4.2. *Let U and V be two relations on X. Suppose that the statistic $\mathrm{maj}'_U+\mathrm{inv}'_V$ is mahonian on X^*. Then, $U\cap V=\emptyset$, $S := U\cup V$ is a total order and a κ-extension of U.*

It is now easy to prove the first part of Theorem 1.8. Indeed, suppose that S is a total order on X and a κ-extension of U. Then, it follows from Theorem 1.6 that $\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U}$ is equidistributed with inv'_S which is mahonian by Lemma 4.1, and thus $\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U}$ is mahonian as desired. Reversely, suppose $\mathrm{maj}'_U+\mathrm{inv}'_V$ is mahonian on X^*. We then have by Lemma 4.2 $V = S\setminus U$ where $S := U\cup V$ is a total order on X and a κ-extension of U.

We thus have proved that the mahonian maj-inv statistics on X^* are exactly those which can be written $\mathrm{maj}'_U+\mathrm{inv}'_{S\setminus U}$, with S a total order on X and a κ-extension of U.

We now prove the second part of Theorem 1.8. Let S and T be two total orders on X and suppose S (resp. T) is a κ-extension of U (resp. V). It suffices to show that if $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}$ and $\mathrm{maj}'_V + \mathrm{inv}'_{T\setminus V}$ are equal on X^* then $S = T$ and $U = V$.

First suppose that $S \neq T$. Then we can assume without loss of generality that there exist $x, y \in X$ such that $x\mathrm{Sy}$ and $x \not{T} y$. Since $U \subseteq S$ and $V \subseteq T$, we then have $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}(xy) = 1 \neq 0 = \mathrm{maj}'_V + \mathrm{inv}'_{T\setminus V}(xy)$ and thus the two statistics are different which contradict the hypothesis, thus $S = T$.

Suppose now $U \neq V$. Then we can assume without loss of generality that there exist $x, y \in X$ such that $x\mathrm{U}y$ and $x \not{V} y$. Since $S = T$ is a total order and an extension of U and V we also have $x\mathrm{Sy}$, $(x, y) \in S \setminus V$ and $y \not{S} y$. It follows that $\mathrm{maj}'_U + \mathrm{inv}'_{S\setminus U}(xy^2) = 1 \neq 2 = \mathrm{maj}'_V + \mathrm{inv}'_{T\setminus V}(xy^2)$, which is impossible thus $U = V$, as desired.

In order to complete the proof of Theorem 1.8, it then remains to prove the two above lemmas.

Proof of Lemma 4.1. It suffices to see that inv'_S is mahonian imply that S is a total order since the reciprocal is an easy consequence of (1.1). Suppose that inv'_S is mahonian, i.e. for each $\mathbf{c}$,

$$\sum_{\mathrm{w}\in\mathcal{R}(\mathbf{c})} q^{\mathrm{inv}'_S(\mathrm{w})} = \begin{bmatrix} c(1) + c(2) + \cdots + c(r) \\ c(1), c(2), \cdots, c(r) \end{bmatrix}_q . \tag{4.1}$$

Suppose there exist $x, y \in X$, $x \neq y$, such that $x \not{S} y$ and $y \not{S} x$. We then have $\mathrm{inv}'_S(xy) = \mathrm{inv}'_S(yx) = 0$, which contradict (4.1) (take $\mathrm{w} = xy$). Thus for each $x, y \in X$, we have $x\mathrm{Sy}$ or $y\mathrm{Sx}$, i.e., S is total.

Suppose there exist $x \in X$ such that $x\mathrm{Sx}$, then $\mathrm{inv}'_S(xx) = 1$, which contradict (4.1) (take $\mathrm{w} = x^2$). Thus $x \not{S} x$ and S is irreflexive.

Suppose there exist $x, y \in X$, $x \neq y$, such that $x\mathrm{Sy}$ and $y\mathrm{Sx}$. We then have $\mathrm{inv}'_S(xy) = \mathrm{inv}'_S(yx) = 1$, which contradict (4.1) (take $\mathrm{w} = xy$). Thus if $x\mathrm{Sy}$ we have $y \not{S} x$, i.e. S is antisymmetric.

Let $x, y, z \in X$ satisfying $x\mathrm{Sy}$ and $y\mathrm{Sz}$. Suppose $x \not{S} z$. Since S is irreflexive, we have $x \neq y$ and $y \neq z$. Since S is antisymmetric, we have $x \neq z$ (otherwise we have $x\mathrm{Sy}$ and $y\mathrm{Sx}$). Then x, y, z are distinct. We also have $y \not{S} x$ and $z \not{S} y$ (S is antisymmetric) and $z\mathrm{Sx}$ (S is total). After simple computations (we left the details to the reader), we then get

$$\sum_{\mathrm{w}\in\mathcal{R}(xyz)} q^{\mathrm{inv}'_S(\mathrm{w})} = 3q + 3q^2 \neq \begin{bmatrix} 3 \\ 1,1,1 \end{bmatrix}_q = 1 + 2q + 2q^2 + q^3,$$

which contradict (4.1) (take $\mathrm{w} = xyz$). Thus $x\mathrm{Sz}$ and S is transitive.

□

Proof of Lemma 4.2. Suppose $U \cap V \neq \emptyset$ and let $(x, y) \in U \cap V$. We then have $\mathrm{maj}'_U(xy) + \mathrm{inv}'_V(xy) = 1 + 1 = 2$, which contradict (4.1) (take $\mathrm{w} = xy$ if $x \neq y$ and $\mathrm{w} = xx$ if $x = y$) and thus, U and V are disjoint.

The proof of "S is a total order on X" is essentially the same than the proof of Lemma 4.1, so we left the details to the reader.

It then remains to show that $S = U \cup V$ is a κ-extension of U. Since S is total, it follows from Lemma 4.1 that inv'_S are mahonian on X^*. Then by applying the part "(i)

imply (ii)" of Theorem 1.6 to $\mathrm{maj}'_U + \mathrm{inv}'_V$ and inv'_S, we obtain that S is a κ-extension of U.

□

5. κ-EXTENSIBLE RELATIONS

Theorem 1.6 and Theorem 1.8 motivate to pose the following question: *When does a relation have a κ-extension?*

Suppose $X = \{x, y, z\}$ and consider the relation $U = \{(x, y), (y, z)\}$ on X. Then, one can check by considering all the relations on X containing U (there are $2^{3^2-2} = 128$ such relations) that U has no κ-extension. In this part, we give an axiomatic characterization of κ-extensible relations.

Definition 5.1. *The κ-closure of a relation U on a set X is the relation denoted by $cl_\kappa(U)$ and defined by*

$$cl_\kappa(U) := U \cup \{(x, y)/ \quad \exists\, z \in X \text{ such that } x\mathrm{U}z \text{ and } y \not\mathrm{U} z\}. \tag{5.1}$$

Proposition 5.2 (Characterization of κ-extensible relations)**.** *Let U be a relation on X. The following conditions are equivalent.*

(i) *U is κ-extensible.*
(ii) *$cl_\kappa(U)$ is a κ-extension of U.*
(iii) *U is transitive and $\nexists\, x, y, z, t \in X$ such that*

$$\begin{array}{ll} x\mathrm{U}y & z \not\mathrm{U} y \\ x \not\mathrm{U} t & z\mathrm{U}t. \end{array} \tag{5.2}$$

For instance, if we consider the relation $U = \{(x, y), (y, z)\}$ on $X = \{x, y, z\}$ given above, we have $x\mathrm{U}y$, $y \not\mathrm{U} y$, $x \not\mathrm{U} z$ and $y\mathrm{U}z$ and thus, we recover that U has no κ-extension. One can also check that the relation "$|$" ("divide") (on $X = [r]$) defined by $x \mid y$ if and only if "x divide y" (i.e. $\frac{y}{x} \in \mathbb{Z}$) has no κ-extension. Indeed, the elements 3,9,2,4 satisfy $3 \mid 9$, $2 \nmid 9$, $3 \nmid 4$ and $2 \mid 4$.

Proof. Clearly (ii) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (iii): Suppose U has a κ-extension S. Then

(a) U is transitive: Indeed, let $x, y, z \in X$ and suppose $x\mathrm{U}y$ and $y\mathrm{U}z$. We want to show that $x\mathrm{U}z$. Suppose $x \not\mathrm{U} z$, then since S is a κ-extension of U and yUz, it follows that ySx and $x \not\mathrm{S} y$. We thus have $x\mathrm{U}y$ and $x \not\mathrm{S} y$, which is impossible since $U \subseteq S$. Thus $x\mathrm{U}z$.
(b) $\nexists\, x, y, z, t \in X$ satisfying $x\mathrm{U}y$, $z \not\mathrm{U} y$, $z\mathrm{U}t$ and $x \not\mathrm{U} t$: Indeed, suppose the contrary. Then, $x\mathrm{U}y$ and $z \not\mathrm{U} y$ imply that xSz, while $z\mathrm{U}t$ and $x \not\mathrm{U} t$ imply that $x \not\mathrm{S} z$. We thus have xSy and $x \not\mathrm{S} y$, which is impossible.

(iii) $\Longrightarrow$ (ii): Suppose U satisfy (iii). We want to show that $H := cl_\kappa(U)$ is a κ-extension of U, that is for any x, y, z satisfying $x\mathrm{U}y$ and $z \not\mathrm{U} y$, we have $x\mathrm{H}z$ and $z \not\mathrm{H} x$. Let x, y, z satisfying $x\mathrm{U}y$ and $z \not\mathrm{U} y$. First, by definition of H, we have $x\mathrm{H}z$. It then remains to show that $z \not\mathrm{H} x$. Suppose the contrary, i.e. $z\mathrm{H}x$. We distinct two cases:

(a) $z\mathrm{U}x$: since U is transitive and $x\mathrm{U}y$, we have $z\mathrm{U}y$, which contradicts $z \not\mathrm{U} y$.

(b) $z \not\mathrm{U} x$ and $z\mathrm{H}x$: by definition of H, there exists t such that $z\mathrm{U}t$ and $x \not\mathrm{U} t$. We thus four elements x, y, z, t satisfying $x\mathrm{U}y$, $z \not\mathrm{U} y$, $x \not\mathrm{U} t$ and $z\mathrm{U}t$, which contradicts (iii). ☐

The following proposition gives some properties of the κ-closure.

Proposition 5.3. *Let U be a κ-extensible relation on X. Then,*

- *$cl_\kappa(U)$ is the smallest κ-extension of U (by inclusion), i.e. every κ-extension of U contains $cl_\kappa(U)$.*
- *$cl_\kappa(U)$ is a bipartitional relation.*

Proof. The first assumption is evident by definition of $cl_\kappa(U)$. Set $H := cl_\kappa(U)$. We claim that H is transitive. Indeed, let $x, y, z \in X$ satisfy $x\mathrm{H}y$ and $y\mathrm{H}z$. We want to show that $x\mathrm{H}z$. We distinct four cases:

(i) $x\mathrm{U}y$, $y\mathrm{U}z$: then, by transitivity of U, we have $x\mathrm{U}z$ and thus $x\mathrm{H}z$ (since $U \subseteq H$).

(ii) $x\mathrm{U}y$, $y \not\mathrm{U} z$ and $y\mathrm{H}z$: then by definition of H, there is $t \in X$ such that $y\mathrm{U}t$ and $z \not\mathrm{U} t$. By transitivity of U, we have $x\mathrm{U}t$. We thus have $x\mathrm{U}t$ and $z \not\mathrm{U} t$, which imply, by definition of H, that $x\mathrm{H}z$.

(iii) $x \not\mathrm{U} y$ and $x\mathrm{H}y$, $y\mathrm{U}z$: then by definition of H, there is $t \in X$ such that $x\mathrm{U}t$ and $y \not\mathrm{U} t$. Suppose $x \not\mathrm{U} z$, then the elements x, t, y, z satisfy $x\mathrm{U}t$, $y \not\mathrm{U} t$, $x \not\mathrm{U} z$, $y\mathrm{U}z$, which contradict (5.2). We thus have $x\mathrm{U}z$, and in particular, $x\mathrm{H}z$.

(iv) $x \not\mathrm{U} y$ and $x\mathrm{H}y$, $y \not\mathrm{U} z$ and $y\mathrm{H}z$: by definition of H, there exist $t, v \in X$ such that $x\mathrm{U}t$, $y \not\mathrm{U} t$, $y\mathrm{U}v$ and $z \not\mathrm{U} v$. Suppose $x \not\mathrm{U} v$, then the elements x, t, y, v satisfy $x\mathrm{U}t$, $y \not\mathrm{U} t$, $x \not\mathrm{U} v$ $y \not\mathrm{U} v$, which contradict (5.2). Thus we have $x\mathrm{U}v$, and since $z \not\mathrm{U} v$, we have by definition of H that xHz.

☐

Let V be a bipartitional relation on X and $(B_1, \ldots, B_k), (\beta_1, \ldots, \beta_k)$ be the bipartition associated to V (see Definition 1.1). Suppose the block B_l consists of the integer $i_1, i_2, \ldots, i_p$. It will be convenient to write $c(B_l)$ for the sequence $c(i_1), c(i_2), \ldots, c(i_p)$ and $m(B_l) = m_l$ for the sum $c(i_1) + c(i_2) + \ldots + c(i_p)$. In particular, $\binom{m_l}{c(B_l)}$ will denote the multinomial coefficient $\binom{c(i_1)+c(i_2)+\ldots+c(i_p)}{c(i_1),c(i_2),\ldots,c(i_p)}$.

Proposition 5.4. *Let U be a κ-extensible relation. It follows that $H := cl_\kappa(U)$ is a bipartitional relation. Let $(B_1, \ldots, B_k), (\beta_1, \ldots, \beta_k)$ be the bipartition associated to H. Then,*

$$\sum_{\mathrm{w} \in \mathcal{R}(\boldsymbol{c})} q^{(\mathrm{maj}'_U + \mathrm{inv}'_{H \setminus U})(\mathrm{w})} = \begin{bmatrix} c(1) + c(2) + \cdots + c(r) \\ m_1, m_2, \cdots, m_k \end{bmatrix}_q \prod_{l=1}^{k} \binom{m_l}{c(B_l)} q^{\beta_l \binom{m_l}{c(B_l)}}. \tag{5.3}$$

More generally, Equation (5.3) *hold for each relation H satisfying (1) H is a κ-extension of U and (2) H is bipartitional on X.*

Proof. It is just a combination of Theorem 1.6, Proposition 5.3 and Proposition 2.1 in [2]. ☐

6. Proof of Theorem 1.9

Theorem 1.8 lead to the following question: *Given a total order S on X, which are the relations U on X such that S is a κ-extension of U?*

Proposition 6.1. *Let U be a relation on X. The following conditions are equivalent.*

(i) *The natural order ">" is a κ-extension of U.*
(ii) *There exists a map $g : X \mapsto X \cup \{\infty\}$ satisfying $g(y) > y$ for each $y \in X$ such that*

$$x\mathrm{U}y \Leftrightarrow x \geq g(y).$$

Moreover, if U satisfy the condition (i), the map g is unique and defined by

$$g(y) = \begin{cases} \min(\{x : x\mathrm{U}y\}), & \text{if } \exists x \text{ such that } x\mathrm{U}y; \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. (i) $\Longrightarrow$ (ii): Suppose ">" is a κ-extension of U and let $y \in X$. Then, define $g(y) \in X \cup \{\infty\}$ by

- $g(y) = \min(\{x; x\mathrm{U}y\})$ if $\exists x \in X$ satisfying $x\mathrm{U}y$,
- $g(y) = \infty$ otherwise.

It is clear that $g(y) > y$ for each $y \in X$ because $U \subseteq " > "$. Let $x, y \in X$. By definition of $g(y)$, we have $x\mathrm{U}y \implies x \geq g(y)$. Now, suppose $x \geq g(y)$. Since $x \in X$, it follows that $g(y) < \infty$ and thus, there exists $z \in X$ such that $z\mathrm{U}y$. We can take $z = g(y)$. Suppose $x \not{U} y$. Since $z\mathrm{U}y$ and $x \not{U} y$ and " > " is a κ-extension of U, we then have $z = g(y) > x$ which contradicts the fact that $x \geq g(y)$. It then follows that $x\mathrm{U}y$. We thus have proved that $x \geq g(y) \implies x\mathrm{U}y$.

(ii) $\Longrightarrow$ (i): Let $x, y, z \in X$ satisfying $x\mathrm{U}y$ and $z \not{U} y$. It follows from (ii) that $x \geq g(y)$ and $z < g(y)$, and thus $x > z$. We thus have proved that " > " is a κ-extension of U. □

The proof of the following result is left to the reader.

Lemma 6.2. *The κ-extensibility on X is transposable by order isomorphism.*

In other words, if S and T are two total orders on X and h is the unique order isomorphism $h : (X, S) \mapsto (X, T)$, i.e h is a permutation of X and $x\mathrm{Sy} \Leftrightarrow \mathrm{h(x)T\,h(y)}$. Then S is a κ-extension of a relation U on X if and only if the total order T is a κ-extension of the relation $V :=$ "$h(U)$" defined by $x\mathrm{V}y \Leftrightarrow h^{-1}(x)\mathrm{U}h^{-1}(y)$.

Combining the above Lemma and Proposition 6.1, we get immediately the following result.

Proposition 6.3. *Let S be a total order on X and U be a relation on X. We denote by f be the (unique) order isomorphism from $(X, " > ")$ to (X, S). The following conditions are equivalent.*

(i) *S is a κ-extension of U.*
(ii) *There exist an unique map $g : X \mapsto X \cup \{\infty\}$ satisfying $g(y) > y$ for each $y \in X$ such that*

$$x\mathrm{U}y \Leftrightarrow f(x) \geq g(f(y)).$$

Clearly, Theorem 1.9 is an immediate consequence of Theorem 1.8 and Proposition 6.3.

7. Applications: new mahonian statistics

In this section, we give some examples of mahonian maj-inv statistics on X^* which can be derived from the results obtained in this paper. Such statistics are entirely characterized in Theorem 1.8 and Theorem 1.9.

Let g_k, $k \in [1, \infty[$, be the maps $X \mapsto X \cup \{\infty\}$ defined for $x \in X$ by

$$g_k(x) = \lfloor kx + 1 \rfloor.\chi(kx < r) + \infty.\chi(kx \geq r)\,.$$

Clearly, for each $x \in X$, we have $g_k(x) > x$. By applying Corollary 1.10 (or Theorem 1.9 with $f = Id$), we obtain immediately the following result.

Proposition 7.1. *The statistics* stat_{g_k}*,* $k \in [1, \infty[$*, defined for* $\mathrm{w} = x_1 x_2 \ldots x_n \in X^*$ *by*

$$\mathrm{stat}_{g_k}(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(\frac{x_i}{x_{i+1}} > k) + \sum_{1 \leq i < j \leq n} \chi(k \geq \frac{x_i}{x_j} > 1)$$

are mahonian on X^**.*

Note that $\mathrm{stat}_{g_1} = \mathrm{inv}$ and $\mathrm{stat}_{g_r} = \mathrm{maj}$. Now for each $B \subseteq X$, let $H_B : X \mapsto X \cup \{\infty\}$ be the map defined for $x \in X$ by $H_B(x) = (x+1).\chi(x \in B,\ x \neq r) + \infty.\chi(x \notin B \text{ or } x = r)$. Since $H_B(x) > x$ for each $x \in X$, we obtain by applying Corollary 1.10 the following result.

Proposition 7.2. *The statistics* stat_{H_B}*,* $B \subseteq X$*, defined for* $\mathrm{w} = x_1 x_2 \ldots x_n \in X^*$ *by*

$$\mathrm{stat}_{H_B}(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(x_i > x_{i+1},\ x_{i+1} \in B) + \sum_{1 \leq i < j \leq n} \chi(x_i > x_j,\ x_j \notin B)$$

are mahonian on X^**.*

For instance, if $B = \{\text{even numbers} \leq r\}$, then the statistic stat_{H_B} defined for words $\mathrm{w} = x_1 x_2 \ldots x_n \in X^*$ by

$$\mathrm{stat}_{H_B}(\mathrm{w}) = \sum_{i=1}^{n-1} i.\chi(x_i > x_{i+1},\ x_{i+1} \text{ is even}) + \sum_{1 \leq i < j \leq n} \chi(x_i > x_j,\ x_j \text{ is odd})$$

is mahonian on X^*.

More generally, for $A, B \subseteq X$, Let $U_{A,B}$ be the relation on X defined by

$$(x, y) \in U_{A,B} \Longleftrightarrow x \in A\ ,\ y \in B \text{ and } x > y\,.$$

Suppose $(x, y), (y, z) \in U_{A,B}$. By definition of $U_{A,B}$, we have $x, y \in A$, $y, z \in B$ and $x > y$ and $y > z$. In particular, $x \in A$, $z \in B$ and $x > z$, i.e. $(x, z) \in U_{A,B}$. It follows that $U_{A,B}$ is transitive. Now suppose there exist $x, y, z, t \in X$ such that $(x, y), (z, t) \in U_{A,B}$ and $(x, t), (z, y) \notin U_{A,B}$. By definition of $U_{A,B}$, we have $x, z \in A$, $y, t \in B$ and $x > y$, $z \leq y$, $x \leq t$, $z > t$. In particular, $x \leq t$ and $x > t$, which is impossible. It then follows from Proposition 5.2 that $U_{A,B}$ is a κ-extensible relation on X. Let $S_{A,B}$ and $S'_{A,B}$ be the relation defined on X by

$$S_{A,B} = \{(x, y) \in X^2 \,|\, x \in A,\ y \notin A\} \cup \{(x, y) \in A^2 \,|\, x > y\}$$
$$S'_{A,B} = S_{A,B} \cup \{(x, y) \in (A^c)^2 \,|\, x > y\}.$$

Then the reader can check that $S'_{A,B}$ and $S_{A,B}$ are two κ-extensions of $U_{A,B}$. Suppose $A=\{a_1,a_2,\ldots,a_k\}_>$. It is then easy to see that $S_{A,B}$ is a bipartitional relation and its associated bipartition is the pair composed by the partition $(\{a_1\},\{a_2\},\ldots,\{a_k\},A^c)$, and the null vector $\mathbf{0}=(0,0,\ldots,0,0)$.

Set $\mathrm{stat}_{A,B}:=\mathrm{maj}'_{U_{A,B}}+\mathrm{inv}'_{S_{A,B}\setminus U_{A,B}}$ and $\mathrm{stat}'_{A,B}:=\mathrm{maj}'_{U_{A,B}}+\mathrm{inv}'_{S'_{A,B}\setminus U_{A,B}}$. By definition, the statistics $\mathrm{stat}_{A,B}$ and $\mathrm{stat}'_{A,B}$ are defined on words $\mathrm{w}=x_1\ldots x_n\in X^*$ by

$$\begin{aligned}\mathrm{stat}_{A,B}(\mathrm{w})&=\sum_{i=1}^{n-1} i.\chi(x_i>x_{i+1},\,x_i\in A,\,x_{i+1}\in B)+\sum_{1\le i<j\le n}\chi(x_i>x_j,\,x_i\in A,\,x_j\in A\setminus B)\\&\quad+\sum_{1\le i<j\le n}\chi(x_i\le x_j,\,x_i\in A,\,x_j\in B\setminus A)+\sum_{1\le i<j\le n}\chi(x_i\in A,\,x_j\notin A\cup B),\\ \mathrm{stat}'_{A,B}(\mathrm{w})&=\mathrm{stat}_{A,B}(\mathrm{w})+\sum_{1\le i<j\le n}\chi(x_i>x_j,\ x_i\,\mathrm{and}\,x_j\notin A).\end{aligned}$$

Applying Theorem 1.8 and Proposition 5.4, we obtain the following result.

Proposition 7.3. *The statistic* $\mathrm{stat}'_{A,B}$ *is mahonian on* X^* *and for each* $\boldsymbol{c}$*,*

$$\sum_{\mathrm{w}\in\mathcal{R}(\boldsymbol{c})} q^{\mathrm{stat}_{A,B}(\mathrm{w})}=\binom{m(A^c)}{c(A^c)}\begin{bmatrix}c(1)+c(2)+\cdots+c(r)\\ c(a_1),c(a_2),\ldots,c(a_k),m(A^c)\end{bmatrix}_q.$$

For instance, if $E=\{$even integers $\le r\}$ and $O=\{$odd integers $\le r\}$, then the statistics $\mathrm{stat}_{E,O}$ and $\mathrm{stat}'_{E,O}$ are defined for $\mathrm{w}=x_1\ldots x_n\in X^*$ by

$$\begin{aligned}\mathrm{stat}_{E,O}(\mathrm{w})&=\sum_{i=1}^{n-1} i.\chi(x_i>x_{i+1},\,x_i\text{ even},\,x_{i+1}\text{ odd})+\sum_{1\le i<j\le n}\chi(x_i>x_j,\,x_i\,\mathrm{and}\,x_j\text{ even})\\&\quad+\sum_{1\le i<j\le n}\chi(x_i\le x_j,\,x_i\,\text{even},\,x_j\text{ odd}),\\ \mathrm{stat}'_{E,O}(\mathrm{w})&=\sum_{i=1}^{n-1} i.\chi(x_i>x_{i+1},\,x_i\text{ even},\,x_{i+1}\text{ odd})+\sum_{1\le i<j\le n}\chi(x_i\le x_j,\,x_i\,\text{even},\,x_j\text{ odd})\\&\quad+\sum_{1\le i<j\le n}\chi(x_i>x_j,\,x_i\,\mathrm{and}\,x_j\text{ have the same parity}).\end{aligned}$$

It then follows from Proposition 7.3 that the statistic $\mathrm{stat}'_{E,O}$ is mahonian and the generating function of $\mathrm{stat}_{E,O}$ on each $\mathcal{R}(\mathbf{c})$ is given by

$$\begin{aligned}\sum_{\mathrm{w}\in\mathcal{R}(\mathbf{c})} q^{\mathrm{stat}_{E,O}(\mathrm{w})}&=\binom{c(1)+c(3)+\ldots+c(2\left\lfloor\frac{r-1}{2}\right\rfloor+1)}{c(1),c(3),\ldots,c(2\left\lfloor\frac{r-1}{2}\right\rfloor+1)}\\&\quad\times\begin{bmatrix}c(1)+c(2)+\cdots+c(r)\\ c(2),c(4),\ldots,c(2\left\lfloor\frac{r}{2}\right\rfloor),c(1)+c(3)+\ldots+c(2\left\lfloor\frac{r-1}{2}\right\rfloor+1)\end{bmatrix}_q.\end{aligned}$$

In particular, if $\mathcal{S}_r$ is the symmetric group of order r, then

$$\sum_{\sigma\in \mathcal{S}_r} q^{\mathrm{stat}_{E,O}(\sigma)} = \left(\left\lfloor \frac{r+1}{2} \right\rfloor\right)! \times \frac{[r]_q!}{[(\lfloor \frac{r+1}{2} \rfloor)]_q!}.$$

Acknowledgments. I am grateful to Jiang Zeng for helpful discussions and to an anonymous referee for a careful reading of the paper.

References

[1] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968), 236–240.
[2] D. Foata and D. Zeilberger, Graphical major indices, J. Comput. Appl. Math. 68 (1996), no. 1-2, 79–101.
[3] D. Foata and C. Krattenthaler, Graphical major indices II, Sém. Lothar. Combin. 34 (1995) (electronic).
[4] G.N. Han, Ordres bipartitionnaires et statistiques sur les mots, (French) [Bipartition orders and statistics on words] The Foata Festschrift. Electron. J. Combin. 3 (1996), no. 2, Research Paper 3, (electronic)
[5] G.N. Han, Une démonstration "vérificative" d'un résultat de Foata-Zeilberger sur les relations bipartitionnaires, J. Computational and Applied Math. 68 (1996), pp. 159-162.
[6] A. Kasraoui and J. Zeng, EulerMahonian statistics on ordered set partitions (II), J. Combin. Theory Ser. A (2008), doi:10.1016/j.jcta.2008.06.012
[7] M. Lothaire, Combinatorics on words. Reading, Addison-Wesley, 1983.
[8] P.A. MacMahon, The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects, Amer. J. Math. 35 (1913), 281–322.
[9] P.A. MacMahon, Combinatory Analysis, vols. 1 and 2, Cambridge Univ. Press, 1915 (reprinted by Chelsea, New York, 1955).
[10] D. Rawlings, The r-major index, J. Combin. Theory Ser. A 31 (1981), no. 2, 175–183.
[11] D. Rawlings, The (q, r)-Simon Newcomb problem, Linear and Multilinear Algebra 10 (1981), no. 3, 253–260.
[12] E. Steingrímsson, Statistics on ordered partitions of sets, preprint, 1999, available at Arxiv:math.CO/0605670.